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Pointing in Real Euclidean Space

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We treat the following problem. Given two vectors of the same length, find an orthogonal transformation that transforms one to the other. This problem arises in many different engineering fields. In particular, it arises in aerospace engineering, where it is called the pointing problem. We establish the theoretical background of the pointing problem in *n* and thus also in three-dimensional space. We give a straightforward solution to this problem, but because it is not unique, we widen the scope of the problem, define the notions of minimal pointing and optimal pointing, and require that the sought matrix be not only orthogonal but also a minimal, or an optimal, pointing. We then give an illustrative solution in three dimensions and then extend the solution to *n* dimensions using two different approaches, which we present and prove. Several examples are given in three and four dimensions. The three-dimensional examples are used to illustrate the characteristics of the solution.

I. Introduction

THERE are cases where a multidimensional vector is given by its components in a certain Cartesian coordinate system, but we wish to perform an orthogonal transformation on the vector in a way that will change its components into some specified values. In particular, we may want to transform the vector such that all its resulting components are equal. Such a case exists, for example, in digital communication (DC). A common class of random signals treated in DC is that of constant-amplitude polynomial-phase signals.[1,2] Recently, a use of polynomial-phase signals has been proposed for DC, by encoding the message symbols in phase parameters.[3] The parameters of these signals can be estimated using a certain representation known as high-order ambiguity function.[2] As is the case with estimators, the estimation of the vector of parameters is performed with a certain accuracy specified by its corresponding covariance matrix. In general, each of the estimated parameters has a different variance. It turns out that the error probability is dominated by the coefficient of the largest variance.[3] Therefore, it is desired to transform the vector of parameters such that they all have equal variance. Consequently, no one is dominant; thereby the error probability is minimized. Another example for such a need exists in coding theory.[3]

Orthogonal transformations are carried out by orthogonal matrices. Orthogonal matrices whose determinant is equal to $+1$ are known as proper orthogonal matrices, and those whose determinant is equal to -1 are called improper orthogonal matrices.[4] Proper orthogonal matrices can express a Cartesian coordinate transformation and thus are also known as rotation matrices or attitude matrices.

If we limit ourselves to proper orthogonal matrices, then the transformation problem just described can be conceived as a rotation of the coordinate system in which the vector is resolved initially. This coordinate rotation is equivalent to a rotation of the vector in the original coordinate system to a direction in which the new vector components meet the desired requirement on its components. We call the latter operation pointing.

Three-dimensional pointing, which is, of course, a special case of the multidimensional problem just presented, is common practice in aerospace guidance problems. Basically, every thrust vectoring, every velocity-vector direction change, and every line-of-sight aiming of devices such as antenna, camera, laser designator, or weapon system, is a three-dimensional pointing problem. Therefore, when restricting ourselves to proper orthogonal matrices, in our treatment of the problem, we will pay special attention to the three-dimensional case and will also use the solution in three dimensions to illustrate the solution in multidimensional spaces.

Not always are we required to perform a proper orthogonal transformation to obtain the desired vector components. Therefore, for the sake of completeness we will not limit ourselves to proper orthogonal matrices. Consequently, we will extend the notion of pointing to include both proper and improper orthogonal matrices but will distinguish between proper pointing and improper pointing, which correspond, respectively, to proper and improper orthogonal matrices.

For simplicity and with no loss of generality, we will assume that the two vectors are of unit length; thus the problem that we wish to solve is posed as follows. Given an n-dimensional unit vector $\boldsymbol{b}$, find an orthogonal transformation T, which transforms $\boldsymbol{b}$ into a given unit vector $\boldsymbol{d}$. Note that because T is orthogonal, it is obvious that $\boldsymbol{d}$ has to have the same length as $\boldsymbol{b}$.

We can solve the problem of pointing in the following manner. Let us write

$$[\boldsymbol{d}D] = \tilde{T}[\boldsymbol{b}B] \tag{1}$$

where D and B are any $n \times (n-1)$ matrices such that $[\boldsymbol{d}D]$ and $[\boldsymbol{b}B]$ are nonsingular and $\tilde{T}$ is some undefined matrix that satisfies Eq. (1). The use of the Gram–Schmidt orthogonalization algorithm on $[\boldsymbol{d}D]$ and $[\boldsymbol{b}B]$ starting with $\boldsymbol{d}$ and $\boldsymbol{b}$ (a numerically reliable way to do that is by using the Householder QR algorithm[5]) transforms the matrices $[\boldsymbol{d}D]$ and $[\boldsymbol{b}B]$ into orthogonal matrices $[\boldsymbol{d}D_0]$ and $[\boldsymbol{b}B_0]$, which satisfy the equation

$$[\boldsymbol{d}D_0] = T[\boldsymbol{b}B_0] \tag{2a}$$

therefore

$$T = [\boldsymbol{d}D_0]\,[\boldsymbol{b}B_0]' \tag{2b}$$

where $'$ denotes the transpose. The matrix T is an orthogonal transformation that transforms $\boldsymbol{b}$ to $\boldsymbol{d}$.

It is quite easy to show that there are infinitely many solutions to this problem unless the vectors $\boldsymbol{d}$ and $\boldsymbol{b}$ are two-dimensional (see Sec. III). Therefore, we are free to select additional requirements on the transformation. In particular, we may require that the transformation be a minimal one or an optimal one. (An exact definition of what constitutes a minimal or an optimal transformation will be given in the ensuing.)

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To formulate precisely the problem of minimal pointings, we need a measure of deviation of a given transformation T from the target transformation that in our case will be the identity matrix I. We use the Frobenius norm $\|A\| = [\mathrm{tr}(AA')]^{1/2}$ to define such a measure. (Another measure of deviation is discussed in Ref. 3.) Thus, the deviation of T from the target transformation is defined by $\|T - I\| = [\mathrm{tr}(T - I)(T - I)']^{1/2}$. We can now state the problems as follows.

II. Statement of the Problem

Given two n-dimensional unit vectors $\boldsymbol{b}$ and $\boldsymbol{d}$.

1) Find the *minimal pointing* T_m, i.e., the orthogonal matrix T_m, such that $T_m\boldsymbol{b} = \boldsymbol{d}$ and $\|T_m - I\|$ is minimal among all deviations of the orthogonal matrices T that satisfy $T\boldsymbol{b} = \boldsymbol{d}$. In other words, find T_m such that

$$\|T_m - I\| \leq \|T - I\|$$

for every orthogonal matrix T that satisfies $T\boldsymbol{b} = \boldsymbol{d}$.

2) Find the *minimal proper pointing* T_m, i.e., the proper orthogonal matrix T_m, such that $T_m\boldsymbol{b} = \boldsymbol{d}$ and $\|T_m - I\|$ is minimal among all deviations of the proper orthogonal matrices T that satisfy $T\boldsymbol{b} = \boldsymbol{d}$.

In what follows, we will give special attention to the three-dimensional case, where the problem and its solution can be easily visualized if the transformation is proper. As will become clear in the following sections, the problem in three dimensions is that of finding an axis around which a single rotation yields the sought solution. This axis, which is known as the Euler axis,[6] is the direction of the eigenvector of the sought transformation matrix T_m whose eigenvalue is 1. Note that the problem treated here is different from that of finding the shortest rotation between two given orientations,[7] because the latter is uniquely determined by the initial and final orientations. Consequently, there exists only one transformation matrix that transforms from the initial to the final orientation, and thus there is only one axis (Euler axis) about which a single rotation rotates the initial to the final orientation (coordinate system).

The optimal pointing problem requires a different measure of deviation: instead of minimal norm, as for minimal pointings, we now ask for the maximal dimension of a subspace on which the pointing coincides with the identity transformation. The exact formulation of the problem is as follows.

3) Find the optimal pointing T_o, i.e., the orthogonal matrix T_o such that $T_o\boldsymbol{b} = \boldsymbol{d}$ and the dimension of the subspace $\{\boldsymbol{x} : T_o\boldsymbol{x} = \boldsymbol{x}\}$ is maximal among all orthogonal matrices T that satisfy $T\boldsymbol{x} = \boldsymbol{x}$.

4) Solve the problem 3, limited to proper orthogonal matrices.

In the next two sections we treat the solution of the problem in two and three dimensions. Then in Sec. V, minimal pointings are described in n dimensions. The solution of the optimal pointing problem is presented in Sec. VI. Comments on the solutions in three dimensions are given in Sec. VII. The work presented in this paper is summarized in Sec. VIII.

The vector spaces in this paper are $\boldsymbol{R}^n$, the vector space of n-dimensional columns with real components, equipped with the standard Euclidean metric (distance and norm).

III. Pointings in Two-Dimensional Space

Because we limited ourselves to unit vectors, $\boldsymbol{b}$ and $\boldsymbol{d}$ can always be written as $\boldsymbol{b}' = [\cos\beta \ \sin\beta]$ and $\boldsymbol{d}' = [\cos\delta \ \sin\delta]$. Following the solution presented in the Introduction, we write

$$T[\boldsymbol{b}\boldsymbol{b}^*] = [\boldsymbol{d}\boldsymbol{d}^*]$$

where $\boldsymbol{d}^*$ and $\boldsymbol{b}^*$ are necessarily two-dimensional columns. To find an orthogonal T, $\boldsymbol{d}^*$ has to be orthogonal to $\boldsymbol{d}$ and $\boldsymbol{b}^*$ has to be orthogonal to $\boldsymbol{b}$. In two dimensions there are precisely four possibilities. If we draw the vectors in the two-dimensional plane, we immediately realize that only the following two unit vectors are orthogonal to $\boldsymbol{d}$:

$$\boldsymbol{d}_{1,2}^{*\prime} = [\mp\sin\delta \ \pm\cos\delta]$$

and only the two following unit vectors are orthogonal to $\boldsymbol{b}$:

$$\boldsymbol{b}_{1,2}^{*\prime} = [\mp\sin\beta \ \pm\cos\beta]$$

therefore we have

$$T\begin{bmatrix}\cos\beta & \mp\sin\beta \\ \sin\beta & \pm\cos\beta\end{bmatrix} = \begin{bmatrix}\cos\delta & \mp\sin\delta \\ \sin\delta & \pm\cos\delta\end{bmatrix}$$

Solving the last equation for T yields

$$T = \begin{bmatrix}\cos\delta & \mp\sin\delta \\ \sin\delta & \pm\cos\delta\end{bmatrix} \cdot \begin{bmatrix}\cos\beta & \sin\beta \\ \mp\sin\beta & \pm\cos\beta\end{bmatrix}$$

and multiplication of the matrices on the right-hand side of the last equation yields only two different matrices. They are

$$T_{\pm} = \begin{bmatrix}\cos(\delta \pm \beta) & \pm\sin(\delta \pm \beta) \\ \sin(\delta \pm \beta) & \mp\cos(\delta \pm \beta)\end{bmatrix}$$

To obtain one of them, the upper sign is taken everywhere; to obtain the other, the lower sign is taken everywhere. Observe that $\det\{T_+\} = -1$, whereas $\det\{T_-\} = 1$. The Frobenius norm of a 2×2 matrix

$$A = \begin{bmatrix}a_{11} & a_{12} \\ a_{21} & a_{22}\end{bmatrix}$$

is $\|A\| = [\mathrm{tr}(AA')]^{1/2} = (a_{11}^2 + a_{12}^2 + a_{21}^2 + a_{22}^2)^{1/2}$. A straightforward computation shows that

$$\|T_+ - I\| = 2 \qquad \text{and} \qquad \|T_- - I\| = 2[1 - \cos(\delta - \beta)]^{\frac{1}{2}}$$

Because there is only one proper pointing, namely, T_-, the minimal proper pointing problem is trivial. A comparison between $\|T_+ - I\|$ and $\|T_- - I\|$ shows that T_- is the minimal pointing if $\cos(\delta - \beta) > 0$, i.e., $\boldsymbol{b}$ and $\boldsymbol{d}$ form an acute angle; T_+ is the minimal pointing if $\boldsymbol{b}$ and $\boldsymbol{d}$ form an obtuse angle $[\cos(\delta - \beta) > 0]$; and both T_+ and T_- are minimal pointings if $\boldsymbol{b}$ and $\boldsymbol{d}$ form a right angle.

IV. Pointings in Three-Dimensional Case

In this section we consider only proper orthogonal transformations. By Euler's theorem,[8] such transformations are accomplished by a single rotation around an axis (called the Euler axis). As mentioned in Sec. II, this axis is determined by the eigenvector corresponding to the eigenvalue 1.

In a three-dimensional space we have one degree of freedom, and thus we have infinitely many orthogonal matrices that will transform $\boldsymbol{b}$ into $\boldsymbol{d}$. However, if we are considering only proper pointings, even though there are still infinite orthogonal matrices that transform $\boldsymbol{b}$ into $\boldsymbol{d}$, there is only one transformation that will be minimal in the sense that it is obtained from a single rotation about a fixed axis by the smallest possible angle, which is the angle between the two vectors. (This will be demonstrated in example 2 of this section.) In Sec. V it will be shown that this matrix is indeed the minimal proper pointing as defined in the Introduction. Let us now demonstrate this point. Similar to the two-dimensional case, the vectors $\boldsymbol{b}$ and $\boldsymbol{d}$ lie in a plane that they define. However, this plane is not necessarily a plane containing two axes of the coordinate system in which $\boldsymbol{b}$ and $\boldsymbol{d}$ are resolved; therefore the rotation is not necessarily about one of the coordinate axes and, therefore, is not necessarily an Euler angle. Nevertheless, finding the minimal rotation is a straightforward procedure. What we have to do is find the angle ϕ from $\boldsymbol{b}$ to $\boldsymbol{d}$, find the normal to the plane defined by the two vectors, and then rotate the coordinate system about this normal by $-\phi$. Let us demonstrate it by the following example.

In this example, as well as in the subsequent discussion, we use the quaternion representation of proper orthogonal matrices, i.e., of rotations. A brief review of this representation follows. Consider the rotation T in the three-dimensional space defined by a unit vector $\hat{\boldsymbol{\Phi}}$ along the axis of rotation and by the angle of rotation ϕ (measured in radians). Let $\boldsymbol{\Phi} = \phi\hat{\boldsymbol{\Phi}}$; the vector $\boldsymbol{\Phi}$ is called the Euler vector of the rotation T. Note that this rotation is uniquely determined by $\boldsymbol{\Phi}$. Let α_x, α_y, α_z be, respectively, the angles between $\hat{\boldsymbol{\Phi}}$ and the coordinate axes x, y, z of the right Cartesian system being rotated by T. Note that

$$\cos\alpha_x = \frac{\phi_x}{\|\boldsymbol{\Phi}\|}, \qquad \cos\alpha_y = \frac{\phi_y}{\|\boldsymbol{\Phi}\|}, \qquad \cos\alpha_z = \frac{\phi_z}{\|\boldsymbol{\Phi}\|}$$

where ϕ_x, ϕ_y, ϕ_z are the components of $\mathbf{\Phi}$ in this system. Now define the quaternion q (called the quaternion of rotation) by

$$q = \cos(\phi/2) + i\sin(\phi/2)\cos\alpha_x + j\sin(\phi/2)\cos\alpha_y + k\sin(\phi/2)\cos\alpha_z$$

where i, j, k are the standard quaternionic units. Observe that $|q| = 1$. Therefore, $q^{-1} = \bar{q}$, where $\bar{q}$ is the conjugate of q. It can be shown that the transformation T can be represented in terms of q as follows: if

$$T\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} \tag{3a}$$

then

$$ix_2 + jy_2 + kz_2 = q^{-1}(ix_1 + jy_1 + kz_1)q \tag{3b}$$

Example 1

Let

$$\boldsymbol{b}' = [0.912871 \quad 0.182574 \quad -0.365148]$$

and

$$\boldsymbol{d}' = [0.57735 \quad 0.57735 \quad 0.57735]$$

The unit vector $\hat{\mathbf{\Phi}}$ in the direction of the normal to the plane defined by $\boldsymbol{b}$ and $\boldsymbol{d}$ (Euler axis) and the angle ϕ are found as follows:

$$\hat{\mathbf{\Phi}} = \frac{\boldsymbol{b}\times\boldsymbol{d}}{|\boldsymbol{b}\times\boldsymbol{d}|}, \qquad \phi = \sin^{-1}\left[\frac{|\boldsymbol{b}\times\boldsymbol{d}|}{|\boldsymbol{b}|\cdot|\boldsymbol{d}|}\right] \tag{4}$$

the results are $\hat{\mathbf{\Phi}}' = [-0.348743\ 0.813733\ -0.464991]$ and $\phi = 1.13555$. Next define the quaternion of rotation, $q = iq_1 + jq_2 + kq_3 + q_4$ for a rotation by $-\phi$ about $\hat{\mathbf{\Phi}}$:

$$q_1 = \sin[-(\phi/2)]\,(\phi_x/\phi), \qquad q_2 = \sin[-(\phi/2)]\,(\phi_y/\phi)$$
$$q_3 = \sin[-(\phi/2)]\,(\phi_z/\phi), \qquad q_4 = \cos[-(\phi/2)] \tag{5}$$

where ϕ_x, ϕ_y, ϕ_z are the components of $\mathbf{\Phi} = \phi\hat{\mathbf{\Phi}}$. Because Eq. (3a) is equivalent to Eq. (3b), that is, both transform the same vector to another vector, there must be a functional relation between the quaternion of rotation and the corresponding transformation matrix. Using this relation,[6] one obtains

$$T_m = \begin{bmatrix} q_1^2 - q_2^2 - q_3^2 + q_4^2 & 2(q_1q_2 + q_3q_4) & 2(q_1q_3 - q_2q_4) \\ 2(q_1q_2 - q_3q_4) & -q_1^2 + q_2^2 - q_3^2 + q_4^2 & 2(q_2q_3 + q_1q_4) \\ 2(q_1q_3 + q_2q_4) & 2(q_2q_3 - q_1q_4) & -q_1^2 - q_2^2 + q_3^2 + q_4^2 \end{bmatrix} \tag{6}$$

When the preceding values of $\hat{\mathbf{\Phi}}$ and ϕ are used in Eq. (5) and the resulting quaternion components are substituted into Eq. (6), one obtains

$$T_m = \begin{bmatrix} 0.491978 & -0.585767 & -0.644076 \\ 0.257507 & 0.804607 & -0.535068 \\ 0.831653 & 0.097388 & 0.546688 \end{bmatrix} \tag{7}$$

An alternative way for finding T_m from the vector $\mathbf{\Phi}$ is as follows. Define the cross-product matrix

$$[\mathbf{\Phi}\times] = \begin{bmatrix} 0 & -\phi_z & \phi_y \\ \phi_z & 0 & -\phi_x \\ -\phi_y & \phi_x & 0 \end{bmatrix} \tag{8}$$

then

$$T_m = \exp\{-[\mathbf{\Phi}\times]\} \tag{9}$$

Because $[\mathbf{\Phi}\times]$ is a skew symmetric matrix, the last expression can also be written as

$$T_m = \cos\phi\cdot I + (1-\cos\phi)\cdot\hat{\phi}\hat{\phi}' - \sin\phi[\mathbf{\Phi}\times] \tag{10}$$

where I is the identity matrix.

As mentioned, it will be shown later that T_m is the minimal proper pointing. Let us check this point for this example. For T_m of the example, we find that

$$\|T_m - I\| = 1.521 \tag{11}$$

The matrix

$$T = \begin{bmatrix} 0.86038 & -0.227924 & 0.455848 \\ 0.36038 & -0.36038 & -0.86038 \\ 0.36038 & 0.904531 & -0.227924 \end{bmatrix}$$

too, is an orthogonal matrix that transforms $\boldsymbol{b}$ to $\boldsymbol{d}$ (T was found using the method described in the Introduction); however,

$$\|T - I\| = 2.33578$$

Thus

$$\|T_m - I\| < \|T - I\|$$

which stems from the fact that among the orthogonal matrices that transform $\boldsymbol{b}$ to $\boldsymbol{d}$, T_m is the closest to I.

In example 1, we found T_m using geometrical considerations; that is, we rotated $\boldsymbol{b}$ into $\boldsymbol{d}$ about an axis normal to the plane in which they lie. We did it using the quaternion of rotation. Because, even for an arbitrary matrix that transforms $\boldsymbol{b}$ into $\boldsymbol{d}$, it is possible to find a corresponding quaternion of rotation, it means that there are other fixed axes of rotation about which we can turn $\boldsymbol{d}$ into $\boldsymbol{b}$ (remember, turning $\boldsymbol{d}$ into $\boldsymbol{b}$ yields a transformation from $\boldsymbol{b}$ to $\boldsymbol{d}$!) using a single rotation. The existence of other axes of rotation that will bring $\boldsymbol{d}$ into $\boldsymbol{b}$ may be hard to conceive. The purpose of the following example is to present a case where it is easy to visualize a rotation that turns $\boldsymbol{d}$ into $\boldsymbol{b}$ but is not minimal, i.e., its corresponding transformation matrix is not minimal proper pointing.

Example 2

In Fig. 1

$$\boldsymbol{b}' = [0 \quad 1 \quad 0]$$

and

$$\boldsymbol{d}' = [0 \quad \cos(\pi/2) \quad \sin(\pi/2)]$$

We realize that $\boldsymbol{b}$ lies along the y axis of a Cartesian coordinate system and each point of $\boldsymbol{d}$ is at an equal distance from the y and z axes. Both vectors are, of course, of equal (unit) length. It is easy to see that S_1, the minimal rotation, is about the axis that lies along the $-x$ axis, and the rotation angle ϑ about this axis is $\pi/2$. Now, there is another legitimate, easily visualized, axis of rotation. It is the

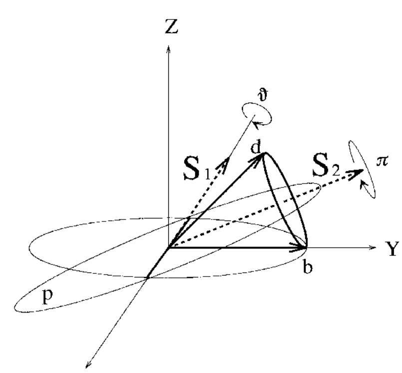

Fig. 1 Geometrical interpretation of the problem in three dimensions.

axis S_2, which lies in the plane defined by $\boldsymbol{b}$ and $\boldsymbol{d}$ (the y–z plane) and bisects the angle between them. It is evident that a rotation about this axis by either π or $-\pi$ brings $\boldsymbol{b}$ into coincidence with $\boldsymbol{d}$, as required. Obviously this angle of rotation is larger than that of the first rotation, which is minimal. These two rotations yield, respectively,

$$T_m = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.707 & -0.707 \\ 0 & 0.707 & 0.707 \end{bmatrix} \tag{12}$$

and

$$T = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0.707 & 0.707 \\ 0 & 0.707 & -0.707 \end{bmatrix} \tag{13}$$

We find that $\|T_m - I\| = 1.082$ and that $\|T - I\| = 2.828$, which verifies again that $\|T_m - I\| < \|T - I\|$. Note that a rotation about any axis lying in the plane p, defined by S_1 and S_2, rotates $\boldsymbol{d}$ into $\boldsymbol{b}$ but only the one about $-x$ is minimal.

After this discussion of the solution in three dimensions, we are now ready to consider the general n-dimensional case.

V. Pointing in n-Dimensional Spaces

Arbitrary Pointing

A construction of an arbitrary pointing in $\boldsymbol{R}^n$ has already been given in the Introduction [see Eq. (2)]. One might elaborate on this as follows. Write

$$\boldsymbol{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}, \qquad \boldsymbol{d} = \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix}$$

Let i and j be indices such that $b_i \neq 0$ and $d_j \neq 0$. Then take in Eq. (1),

$$B = [\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_{i-1}, \boldsymbol{e}_{i+1}, \ldots, \boldsymbol{e}_n]$$

and

$$D = [\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_{j-1}, \boldsymbol{e}_{j+1}, \ldots, \boldsymbol{e}_n]$$

where $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$ are the columns of the $n \times n$ identity matrix.

To reduce the numerical errors in the subsequent Gram–Schmidt reduction, we propose to use the indices i and j determined by the properties that

$$b_i = \max(|b_1|, |b_2|, \ldots, |b_n|)$$

$$|d_j| = \max(|d_1|, |d_2|, \ldots, |d_n|)$$

Minimal Pointing

Let there be given two unit vectors $\boldsymbol{b}$ and $\boldsymbol{d}$ in $\boldsymbol{R}^n$. Recall that an orthogonal $n \times n$ matrix T is called pointing if $T\boldsymbol{b} = \boldsymbol{d}$. A pointing T_m is called minimal if the norm

$$\|T_m - I\| = [\mathrm{tr}(T_m - I)(T_m - I)']^{\frac{1}{2}}$$

is minimal among all pointings T. We denote by $\langle x, y\rangle$ the standard scalar product of the vectors

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \quad \text{and} \quad y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

where

$$\langle x, y\rangle = \sum_{j=1}^{n} x_j y_j$$

The solution of the minimal pointing problem is based on the following theorem.

Theorem 1. Let $\boldsymbol{b}$ and $\boldsymbol{d}$ be unit vectors in $\boldsymbol{R}^n$.

a) The minimal deviation among all pointings,

$$Q = \min_T \|T - I\|$$

subject to $TT' = I$, $T\boldsymbol{b} = \boldsymbol{d}$, is given by the formula

$$Q = \begin{cases} 2(1 - \langle \boldsymbol{d}, \boldsymbol{b}\rangle)^{\frac{1}{2}} & \text{if} \quad \langle \boldsymbol{d}, \boldsymbol{b}\rangle > 0 \\ 2 & \text{if} \quad \langle \boldsymbol{d}, \boldsymbol{b}\rangle \leq 0 \end{cases} \tag{14}$$

b) If $\langle \boldsymbol{d}, \boldsymbol{b}\rangle \neq 0$ and $\boldsymbol{d} \neq \pm\boldsymbol{b}$, then the minimal pointing T_m is unique and is uniquely determined by the following Eqs. (15) and (16), where $\boldsymbol{c}$ is a fixed unit vector that is a linear combination of $\boldsymbol{b}$ and $\boldsymbol{d}$ and that is orthogonal to $\boldsymbol{b}$:

$$T_m\boldsymbol{b} = \boldsymbol{d}, \qquad T_m\boldsymbol{c} = \pm(-\langle \boldsymbol{d}, \boldsymbol{c}\rangle\boldsymbol{b} + \langle \boldsymbol{d}, \boldsymbol{b}\rangle\boldsymbol{c}) \tag{15}$$

the sign in Eq. (15) is $+1$ if $\langle \boldsymbol{d}, \boldsymbol{b}\rangle > 0$, and -1 if $\langle \boldsymbol{d}, \boldsymbol{b}\rangle < 0$;

$$T_m\boldsymbol{x} = \boldsymbol{x} \quad \text{if} \quad \boldsymbol{x} \perp \boldsymbol{b}, \boldsymbol{x} \perp \boldsymbol{d} \tag{16}$$

c) If $\boldsymbol{d} = \boldsymbol{b}$, then I is the unique minimal pointing. If $\boldsymbol{d} = -\boldsymbol{b}$, then the unique minimal pointing T_m is defined by the properties

$$T_m\boldsymbol{b} = -\boldsymbol{b}, \qquad T_m\boldsymbol{x} = \boldsymbol{x} \quad \text{for every} \quad \boldsymbol{x} \perp \boldsymbol{b} \tag{17}$$

If $\langle \boldsymbol{d}, \boldsymbol{b}\rangle = 0$, then there are exactly two minimal pointings T_m given by

$$T_m\boldsymbol{b} = \boldsymbol{d}, \qquad T_m\boldsymbol{d} = \pm\boldsymbol{b}, \qquad T_m\boldsymbol{x} = \boldsymbol{x} \quad \text{for all} \quad \boldsymbol{x} \tag{18}$$

that are orthogonal to both $\boldsymbol{b}$ and $\boldsymbol{d}$.

For the proof of Theorem 1, the reader is referred to Ref. 3.

An algorithm for computing the optimal pointing is proposed, as follows. We assume that $\boldsymbol{d} \neq \pm\boldsymbol{b}$ and $\langle \boldsymbol{d}, \boldsymbol{b}\rangle \neq 0$. Let $\phi_1, \ldots, \phi_{n-2}$ be n-dimensional vectors such that

$$\boldsymbol{b}, (\boldsymbol{d} - \langle \boldsymbol{d}, \boldsymbol{b}\rangle\boldsymbol{b})\|\boldsymbol{d} - \langle \boldsymbol{d}, \boldsymbol{b}\rangle\boldsymbol{b}\|^{-1}, \phi_1, \ldots, \phi_{n-2} \tag{19}$$

form an orthonormal basis in $\boldsymbol{R}^n$ (to obtain such vectors, apply the Gram–Schmidt orthonormalization to a linearly independent set $\boldsymbol{b}, \boldsymbol{d}, \Psi_1, \ldots, \Psi_{n-2}$, where, for example, $\Psi_1, \ldots, \Psi_{n-2}$ can be unit coordinate vectors, suitably chosen). Let A be the orthogonal $n \times n$ matrix having the columns of Eq. (19) (in this order). Then

$$T_o = A \begin{bmatrix} \langle \boldsymbol{d}, \boldsymbol{b}\rangle & \pm(-q) & 0 \\ q & \pm\langle \boldsymbol{d}, \boldsymbol{b}\rangle & 0 \\ 0 & 0 & I_{n-2} \end{bmatrix} A' \tag{20}$$

where $q = (1 - \langle \boldsymbol{d}, \boldsymbol{b}\rangle^2)\|\boldsymbol{d} - \langle \boldsymbol{d}, \boldsymbol{b}\rangle\boldsymbol{b}\|^{-1}$ and the sign in Eq. (20) is $+$ if $\langle \boldsymbol{d}, \boldsymbol{b}\rangle > 0$ and $-$ if $\langle \boldsymbol{d}, \boldsymbol{b}\rangle < 0$.

Minimal Proper Pointing

Recall that a pointing T is called proper if $\det(T) = 1$. We now present a solution to the minimal proper pointing problem (problem 2 of the Introduction).

Theorem 2. Let $\boldsymbol{b}$ and $\boldsymbol{d}$ be unit vectors in $\boldsymbol{R}^n$.

a) The minimal deviation among all proper pointings,

$$Q_p = \min_T \|T - I\|$$

subject to $TT' = I$, $T\boldsymbol{b} = \boldsymbol{d}$, $\det(T) = 1$ is given by the formula

$$Q_p = 2(1 - \langle \boldsymbol{d}, \boldsymbol{b}\rangle)^{\frac{1}{2}}$$

b) If $\boldsymbol{d} \neq -\boldsymbol{b}$, then a minimal proper pointing, i.e., a proper pointing T_m such that $\|T_m - I\| = 2(1 - \langle \boldsymbol{d}, \boldsymbol{b}\rangle)^{1/2}$, is unique and is given by the rotation that transforms $\boldsymbol{b}$ to $\boldsymbol{d}$ and leaves every vector in the orthogonal complement to span$\{\boldsymbol{b}, \boldsymbol{d}\}$ unchanged.

c) Assume $\boldsymbol{d} = -\boldsymbol{b}$. Then all minimal proper pointings T_m are given by the following recipe. Select any unit vector $\boldsymbol{v}$ that is orthogonal to $\boldsymbol{b}$, and define T_m as the rotation through the angle π that transforms $\boldsymbol{b}$ to $\boldsymbol{d}$, and $\boldsymbol{v}$ to $-\boldsymbol{v}$ and leaves every vector in the orthogonal complement to span$\{\boldsymbol{b}, \boldsymbol{v}\}$ unchanged.

The result of Theorem 2 in case $\langle \boldsymbol{d}, \boldsymbol{b}\rangle \geq 0$ is essentially contained in Theorem 1. Indeed, the formula in Eq. (15) with the sign $+1$ gives the rotation described in part b of Theorem 2. Because it is the minimal pointing by Theorem 1, it is also the minimal proper pointing.

We will give independent proof of Theorem 2, which is applicable to all cases. First, two lemmas are needed, the proofs of which can be found in Ref. 3.

Lemma 1. Let $z \in (0, \pi)$. The function

$$F(x_1, \ldots, x_q) = x_1 + \cdots + x_q$$

subject to the following conditions:

$$0 \leq x_j \leq 2, \qquad p_j \geq 0, \qquad p_1 + \cdots + p_q \leq 1 \tag{21}$$

$$\sum_{j=1}^{q} \arccos(1 - p_j x_j) \geq z \tag{22}$$

attains its minimal value when all but one of the variables among $x_1, \ldots, x_q$ are equal to zero and the remaining variable is equal to $1 - \cos z$. Obviously, the minimal value of F is then equal to $1 - \cos z$.

The values of the arccos function are assumed to be in $[0, \pi]$.

Lemma 2. Let T_α be an elementary rotation in $\boldsymbol{R}^n$ that rotates a pair of unit length orthogonal vectors $\boldsymbol{u}, \boldsymbol{v}$ through the angle $\alpha (0 \leq \alpha \leq \pi)$ and leaves every vector orthogonal to span$\{\boldsymbol{u}, \boldsymbol{v}\}$ unchanged; that is, $T_\alpha \boldsymbol{y} = \boldsymbol{y}$ if $\boldsymbol{y} \perp \text{span}\{\boldsymbol{u}, \boldsymbol{v}\}$. Then for every unit length vector $\boldsymbol{c} \in \boldsymbol{R}^n$, the angle between $\boldsymbol{c}$ and $T_\alpha \boldsymbol{c}$ is equal to $\arccos[1 - p^2(1 - \cos\alpha)]$, where p is the length of the orthogonal projection of $\boldsymbol{c}$ onto span$\{\boldsymbol{u}, \boldsymbol{v}\}$.

Proof of Theorem 2. We treat first the relatively easy part to prove, part c. Thus, assume $\boldsymbol{d} = -\boldsymbol{b}$. Clearly, for every pointing T the one-dimensional subspace span$\{\boldsymbol{b}\}$ is invariant. Hence $W = \text{span}\{\boldsymbol{b}\}^\perp$, the orthogonal complement of span$\{\boldsymbol{b}\}$, is also T-invariant. Now T induces an orthogonal transformation on W [which will be denoted $T(W)$], and the determinant of $T(W)$ is -1. Thus, T is a minimal proper pointing if and only if the norm $\|I - T(W)\|$ is minimal among all orthogonal transformations on W with determinant -1. It is easy to see that the orthogonal transformation Y on W with determinant -1 for which the norm $\|I - Y\|$ is minimal are of the form $\boldsymbol{Y}\boldsymbol{v} = -\boldsymbol{v}, \boldsymbol{Y}\boldsymbol{w} = \boldsymbol{w}$ for all $\boldsymbol{w} \perp \boldsymbol{v}$, where $\boldsymbol{v} \in W$ is any nonzero vector. Indeed, any orthogonal transformation Y on W with determinant -1 must have an eigenvector $\boldsymbol{v}$ of unit length corresponding to the eigenvalue -1. Representing Y as a matrix in an orthonormal basis that has $\boldsymbol{v}$ as one of the vectors in the basis, we see that $\|I - Y\| \geq 2$. To ensure that this norm is minimal, the transformation Y must have the form indicated earlier. This proves part c of Theorem 2.

For the proof of parts a and b, we will use the notion of canonical angles. Let T be a proper orthogonal matrix. It is well known[9] that there exists an orthogonal matrix U such that UTU' has the form

$$UTU' = \begin{bmatrix} \cos\alpha_1 & -\sin\alpha_1 \\ \sin\alpha_1 & \cos\alpha_1 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} \cos\alpha_q & -\sin\alpha_q \\ \sin\alpha_q & \cos\alpha_q \end{bmatrix} \oplus I \tag{23}$$

where $\alpha_1, \ldots, \alpha_q \in (0, \pi]$ and the identity matrix is $(n - 2k) \times (n - 2k)$; we use here the notation $Z_1 \oplus Z_2 \oplus \cdots \oplus Z_k$ to denote the block diagonal matrix with the diagonal blocks $Z_1, Z_2, \ldots, Z_k$ (in this order). The angles $\alpha_1, \ldots, \alpha_q$ are uniquely determined by T and are called the canonical angles. The deviation of T can be easily expressed in terms of the canonical angles. Namely, because

$$\|T - I\| = \|U(T - I)U'\| = \|UTU' - I\|$$

using the right-hand side of Eq. (23) in place of UTU', we obtain

$$\|T - I\|^2 = 4 \sum_{j=1}^{q} (1 - \cos\alpha_j)$$

Another useful consequence of Eq. (23) is that every proper orthogonal matrix can be represented as a composition of at most $n/2$ elementary rotations $T_{\alpha_1}, \ldots, T_{\alpha_q}$ (which commute, i.e., $T_{\alpha_j} T_{\alpha_k} = T_{\alpha_k} T_{\alpha_j}$). Indeed, let $\boldsymbol{u}_1, \boldsymbol{v}_1, \boldsymbol{u}_2, \boldsymbol{v}_2, \ldots, \boldsymbol{u}_q, \boldsymbol{v}_q, \ldots$ be the columns of U' (in this order). Then clearly the vectors $\boldsymbol{u}_1, \boldsymbol{v}_1, \ldots, \boldsymbol{u}_q, \boldsymbol{v}_q$ form an orthonormal set, and T_{α_j} is the counterclockwise rotation through the angle α_j in the two-dimensional plane span$\{\boldsymbol{u}_j; \boldsymbol{v}_j\}$, whereas every vector orthogonal to span$\{\boldsymbol{u}_j; \boldsymbol{v}_j\}$ is unaltered by T_{α_j}.

Now let T be a proper pointing; thus, $T\boldsymbol{b} = \boldsymbol{d}$. Let $\boldsymbol{b}_1 = T_{\alpha_1}\boldsymbol{b}$; $\boldsymbol{b}_2 = T_{\alpha_2}\boldsymbol{b}_1; \ldots; \boldsymbol{d} = \boldsymbol{b}_q = T_{\alpha_q}\boldsymbol{b}_{q-1}$, where $T_{\alpha_1}, \ldots, T_{\alpha_q}$ are the elementary rotations introduced earlier. Put formally, $\boldsymbol{b}_0 = \boldsymbol{b}$. By Lemma 2 the angle between $\boldsymbol{b}_{j-1}$ and $\boldsymbol{b}_j$ is $\arccos[1 - p_j^2(1 - \cos\alpha_j)]$, where p_j is the length of the orthogonal projection of $\boldsymbol{b}_{j-1}$ onto span$\{\boldsymbol{u}_j, \boldsymbol{v}_j\}$. We obviously have

$$\sum_{j=1}^{q} \arccos[1 - p_j^2(1 - \cos\alpha_j)] \geq z$$

where $z \in (0, \pi)$ is the angle between $\boldsymbol{d}$ and $\boldsymbol{b}$ (we leave aside the trivial case when $\boldsymbol{d} = \boldsymbol{b}$ and the case when $\boldsymbol{d} = -\boldsymbol{b}$, which is taken care of by part c of Theorem 2). Because every vector in span$\{\boldsymbol{u}_j, \boldsymbol{v}_j\}$ is unchanged by $T_{\alpha_1}, \ldots, T_{\alpha_{j-1}}$, the orthogonal projection of $\boldsymbol{b}_{j-1}$ onto span$\{\boldsymbol{u}_j, \boldsymbol{v}_j\}$ coincides with the orthogonal projection of $\boldsymbol{b}$ onto span$\{\boldsymbol{u}_j, \boldsymbol{v}_j\}$. It follows that

$$p_1^2 + \cdots + p_q^2 \leq 1$$

because the length of $\boldsymbol{b}$ is 1. Now application of Lemma 1 (with $x_1 = 1 - \cos\alpha_1, \ldots, x_q = 1 - \cos\alpha_q$) guarantees that a minimal proper pointing must have only one canonical angle, and this angle is equal to z.

This concludes the proof of Theorem 2.

VI. Optimal Pointing

In this section we suggest a different mathematical approach to select a pointing in the set of all pointings. Namely, the selection will be based on the maximal dimension of the subspace consisting of the vectors unaltered by a pointing.

As before, let $\boldsymbol{b}$ and $\boldsymbol{d}$ be fixed unit length vectors in $\boldsymbol{R}^n$. As defined in the preceding statement of the problem in the Introduction, a pointing T is said to be optimal if the subspace of all vectors $\boldsymbol{x}$ that are invariant under T, i.e., $T\boldsymbol{x} = \boldsymbol{x}$, has maximal dimension among all pointings.

It will be assumed throughout this section that $\boldsymbol{b} \neq \boldsymbol{d}$.

Theorem 3. The reflection with respect to span$\{\boldsymbol{b} + \boldsymbol{d}, W\}$ is the (only) optimal pointing.

See Ref. 3 for the proof of Theorem 3.

Theorem 3 provides a simple procedure for the computation of an optimal pointing as is demonstrated by the following example.

Example 3

Let

$$\boldsymbol{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{d} = \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}$$

By solving the system $\langle \boldsymbol{b}, \boldsymbol{x}\rangle = 0$, $\langle \boldsymbol{d}, \boldsymbol{x}\rangle = 0$, we determine that the vectors

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{v}_2 = \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

form a basis for W. We have

$$\boldsymbol{b} + \boldsymbol{d} = \begin{bmatrix} 5 \\ 5 \\ 5 \\ 5 \end{bmatrix}$$

The optimal pointing T_o is defined by $T_o v_1 = v_1$, $T_o v_2 = v_2$, $T_o(\boldsymbol{b} + \boldsymbol{d}) = \boldsymbol{b} + \boldsymbol{d}$, and $T_o \boldsymbol{b} = \boldsymbol{d}$. Therefore

$$T_o \begin{bmatrix} 1 & 2 & 5 & 1 \\ -2 & -3 & 5 & 2 \\ 1 & 0 & 5 & 3 \\ 0 & 1 & 5 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 5 & 4 \\ -2 & -3 & 5 & 3 \\ 1 & 0 & 5 & 2 \\ 0 & 1 & 5 & 1 \end{bmatrix} \Rightarrow T_o$$

$$= \begin{bmatrix} 0.1 & -0.3 & 0.3 & 0.9 \\ -0.3 & 0.9 & 0.1 & 0.3 \\ 0.3 & 0.1 & 0.9 & -0.3 \\ 0.9 & 0.3 & -0.3 & 0.1 \end{bmatrix}$$

A comparison between Theorems 1 and 3 reveals that, if $\langle \boldsymbol{d}, \boldsymbol{b} \rangle \neq 0$, then the minimal and optimal pointings coincide. In other words, the pointing that has the least deviation from the identity matrix is also the pointing for which the dimension of the subspace of invariant vectors under the transformation T_o is maximal. In the case $\langle \boldsymbol{d}, \boldsymbol{b} \rangle = 0$, among the two minimal pointings T_o given by Eq. (18), only one is optimal, namely, the pointing T_o defined by $T_o \boldsymbol{b} = \boldsymbol{d}$; $T_o \boldsymbol{d} = \boldsymbol{b}$; $T_o \boldsymbol{x} = \boldsymbol{x}$ for all $\boldsymbol{x} \in W$.

Consider now optimal proper pointings, i.e., proper pointings T_o such that the subspace of vectors unaltered by T_o has maximal dimension among all proper pointings.

Theorem 4. A pointing T_o is an optimal proper pointing if and only if it is a rotation around any $n - 2$ dimensional subspace of span$\{\boldsymbol{b} + \boldsymbol{d}, W\}$ that transforms $\boldsymbol{b}$ to $\boldsymbol{d}$.

The proof of Theorem 4 is found in Ref. 3.

In view of Theorem 4, we can now introduce the following definition. A pointing T_o is said to be a perfect optimal proper pointing if it is an optimal proper pointing with minimum rotation angle.

As is observed earlier, if $\boldsymbol{b} + \boldsymbol{d} \neq 0$, then $\boldsymbol{b}$ and $\boldsymbol{d}$ are linearly independent and $\dim(W) = n - 2$. In this case we have the following theorem.

Theorem 5. If $\boldsymbol{b} \neq -\boldsymbol{d}$, then the rotation around W that rotates $\boldsymbol{d}$ to $\boldsymbol{b}$ (yielding a transformation from $\boldsymbol{b}$ to $\boldsymbol{d}$, see the paragraph before example 2 and the third paragraph in Sec. I) is the (only) perfect optimal proper pointing.

Proof. As is well known, the rotation angle that rotates $\boldsymbol{d}$ to $\boldsymbol{b}$ is greater than or equal to the angle between $\boldsymbol{b}$ and $\boldsymbol{d}$ (e.g., see example 2 and the corresponding Fig. 1). The latter is the rotation angle if we rotate span$\{\boldsymbol{b}, \boldsymbol{d}\}$ around span$\{\boldsymbol{b}, \boldsymbol{d}\}^{\perp} = W$.

A comparison with Theorem 2 shows that (if $\boldsymbol{b} \neq -\boldsymbol{d}$) the perfect optimal proper pointing coincides with the minimal proper pointing. Theorem 5 provides a simple procedure for the computation of a perfect optimal pointing, as is demonstrated by the following example.

Example 4

As in example 3, let

$$\boldsymbol{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \quad \text{and} \quad \boldsymbol{d} = \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}$$

The vectors

$$v_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad v_2 = \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

form a basis for W. Our rotation T satisfies $T v_1 = v_1$ and $T v_2 = v_2$. Because T is an orthogonal transformation, it maps a vector orthogonal to v_1, v_2, and $\boldsymbol{b}$ to a vector orthogonal to v_1, v_2, and $\boldsymbol{d}$. Therefore, the vector

$$v_3 = \begin{bmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} & \boldsymbol{l} \\ 1 & 2 & 3 & 4 \\ 1 & -2 & 1 & 0 \\ 2 & -3 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 20 \\ 10 \\ 0 \\ -10 \end{bmatrix}$$

is mapped on

$$v_4 = \begin{bmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} & \boldsymbol{l} \\ 4 & 3 & 2 & 1 \\ 1 & -2 & 1 & 0 \\ 2 & -3 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 10 \\ -10 \\ -20 \end{bmatrix}$$

The perfect optimal proper pointing T_o is thus defined by $T_o v_1 = v_1$, $T_o v_2 = v_2$, $T_o \boldsymbol{b} = \boldsymbol{d}$, and $T_o v_3 = v_4$, and hence

$$T_o \begin{bmatrix} 1 & 2 & 1 & 20 \\ -2 & -3 & 2 & 10 \\ 1 & 0 & 3 & 0 \\ 0 & 1 & 4 & -10 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 & 10 \\ -2 & -3 & 3 & 0 \\ 1 & 0 & 2 & -10 \\ 0 & 1 & 1 & -20 \end{bmatrix} \Rightarrow T_o$$

$$= \frac{1}{30} \begin{bmatrix} 23 & 1 & 9 & 17 \\ -9 & 27 & 3 & 9 \\ -11 & -7 & 27 & 1 \\ -13 & -11 & -9 & 23 \end{bmatrix}$$

VII. Comments

In Sec. IV we discussed in some length the practical use of pointing in three dimensions where, if the transformation is a proper one, the problem and its solution are easily visualized. Accordingly, in our geometric approach to the problem, we were looking for the axis of rotation about which the rotation of $\boldsymbol{b}$ into $\boldsymbol{d}$ was minimal; that is, the rotation angle was minimal. Obviously, this axis is the normal to the plane in which $\boldsymbol{d}$ and $\boldsymbol{b}$ lie, and the rotation angle is the angle between $\boldsymbol{d}$ and $\boldsymbol{b}$. We found the corresponding transformation matrix T_m using the quaternion of rotation. It is easy to see that we were finding the perfect optimal proper pointing (POPP) in three dimensions (see its definition in n dimensions). It is not difficult to see that this minimal matrix T_m is also the minimal proper pointing (MPP) in three dimensions. Let us now see, geometrically, how the problem solved in three dimensions corresponds to POPP and MPP. Before doing so, let us be reminded that the minimal rotation problem in three dimensions, the POPP and the MPP, all deal with proper transformations. Bearing in mind that this is the case, we omit, in the following discussion, the specific reference to proper rotations.

As shown in Sec. VI, barring the trivial case when $\boldsymbol{d}$ and $\boldsymbol{b}$ are orthogonal, for the POPP in n dimensions, the dimension of the subspace $W = \text{span}\{\boldsymbol{b}, \boldsymbol{d}\}^{\perp}$ is $n - 2$, and the rotation angle about W that brings $\boldsymbol{b}$ into $\boldsymbol{d}$ is minimal. Moreover, then the MPP and POPP are identical. Now, in three dimensions, span$\{\boldsymbol{b}, \boldsymbol{d}\}$ is the plane defined by $\boldsymbol{b}$ and $\boldsymbol{d}$, the dimension of W is, of course, 1, and W is no other than the Euler axis about which the rotation is performed. The angle of rotation is, of course, the angle between $\boldsymbol{d}$ and $\boldsymbol{b}$. The correspondence between the MPP and the n-dimensional case is the same, only that Theorem 1 treats the problem in a different basis. In the three-dimensional case this means that the Cartesian coordinates are defined such that $\boldsymbol{d}$ and $\boldsymbol{b}$ lie in the x–y plane, and thus W, or the rotation axis, is along the z axis. Thus, I_{n-2} in Eq. (20) is just the number 1.

VIII. Conclusions

This paper lays out the theoretical background for the three-dimensional minimal (optimal) pointing problem, which is encountered in target acquisition, target designation, thrust vectoring, robotics, and guidance problems in general. This problem is also encountered in coding theory and information theory, although there, the pointing is not necessarily proper pointing.

The problem addressed in the paper is that of finding an orthogonal transformation of a given vector $\boldsymbol{b}$ in $\boldsymbol{R}^n$ to another given vector $\boldsymbol{d}$ of same length in $\boldsymbol{R}^n$. We solved the general case of such transformations where the transformation matrix could be either proper or improper, and in addition certain optimality constraints are imposed. In n dimensions (where $n > 2$), we obviously have infinitely many such transformations. If we impose some constraints on the transformation, we limit the number of solutions. One such constraint is imposed by our search for the minimal pointing. Another constraint is imposed by our search for optimal pointing. When we limit these special transformations to proper transformations, then we obtain minimal proper pointing and optimal proper pointing, respectively. If we further limit the latter to be of minimum rotation angle, then we obtain what we define as perfect optimal proper pointing. Finally, it was shown that the MPP and the POPP are identical.

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